Physics 566: Quantum Optics I

Problem Set 1
Due Tuesday, September 2, 2013

Problem 1: Weiner-Khintchine Theorem (10 Points)
Consider a real function of t that is a random variable, $f(t)$, with stationary statistics, i.e., all correlation functions depend only on the time-differences. Thus, the autocorrelation function satisfies $\langle f(t_1)f(t_2)\rangle = \langle f(0)f(t_2-t_1)\rangle = K(t_2-t_1)$.

(a) Defining the Fourier transform in our usual way, $\tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{+i\omega t}\,dt$, show that

$$\langle \tilde{f}^*(\omega)\tilde{f}(\omega')\rangle = 2\pi S(\omega)\delta(\omega-\omega'),$$

where $S(\omega) = \int_{-\infty}^{\infty} K(\tau)e^{+i\omega\tau}\,d\tau$ is the Fourier transform of the autocorrelation function.

The important take-away messages are:

- The frequency components of stationary random field are uncorrelated with one another.
- The function $S(\omega)$ is known as the *spectral density*. $S(\omega)d\omega$ is the amount of "power" in a given spectral band $d\omega$ around ω.
- For a stationary field, the spectral density and the autocorrelation function are Fourier transform pairs. This is the *Weiner-Khintchine Theorem.*

(b) Consider the complex correlation function that determines temporal coherence in a standard interferometer $\Gamma(\tau) = \langle \tilde{E}^*(0)\tilde{E}(\tau)\rangle$, where $\tilde{E}(t)$ is the complex analytic signal, $E(t) = \mathrm{Re}\left[\tilde{E}(t)\right]$. Show that

$$\frac{1}{2}\mathrm{Re}\left[\Gamma(\tau)\right] = \int_{-\infty}^{\infty} S(\omega)e^{-i\omega\tau}\frac{d\omega}{2\pi}, \quad S(\omega) = \int_{-\infty}^{\infty} \frac{1}{2}\mathrm{Re}\left[\Gamma(\tau)\right]e^{+i\omega\tau}\,d\tau,$$

where $\langle \tilde{E}^*(\omega)\tilde{E}(\omega')\rangle = 2\pi S(\omega)\delta(\omega-\omega')$.

Take away message:

- The temporal coherence of a (stationary) field as measured in a standard interferometer is determined by the spectrum of the input field via a Fourier conjugate pair. A very narrow-band (monochromatic) field has a long coherence time, and a broad-band field has a short temporal coherence time.

(c) What is the power spectrum of natural light arising from a collision broadened source? Sketch the output intensity from a Mach-Zender interferometer.

Problem 2: Natural Light (15 Points)
As discussed in Lecture 2, natural light arising from, e.g. stars, is not coherent. The phase of the field fluctuates, and is only well correlated for a short ``coherence time." One source of those fluctuations is random collisions between the radiators. Let's fill in some of the details

(a) Let $P_s(t)$ be the "survival probability," i.e., the probability that the molecule freely oscillates and survives a time t without a collision. Under the assumption that the time of the next collision is independent of the previous (such as random process is said to be *Markovian* – there is no "memory" of the previous trajectory), show that

$$P_s(t) = e^{-\gamma t}\text{, where } \gamma \text{ is the rate of collisions,}$$

(b) Show that the probability that the oscillator free oscillates for time t and then suffers a collision between times t and $t+dt$ is

$p(t)dt = e^{-t/\tau_0}\dfrac{dt}{\tau_0}$, where τ is the average time between collisions. Express τ in terms if γ.

Use the kinetic theory of gases to show that $\dfrac{1}{\tau_0} = n\sigma_0 \bar{v}_{rel}$, where n is the density of molecules, σ_0 is the collision cross section and $\bar{v}_{rel}$ is the average relative speed of the molecules. What is $\bar{v}_{rel}$ for a gas in thermal equilibrium?

(b) The electric field produced by each of the oscillators will has a random phase, $E_i(t) = E_0 e^{-i\omega t} e^{i\phi_i(t)}$.
The total field $E(t) = \sum_{i=1}^{N} E_0 e^{-i\omega t} e^{i\phi_i(t)} = E_0 e^{-i\omega t}\alpha(t) = E_0 e^{-i\omega t} a(t) e^{i\varphi(t)}$, where $\alpha(t)$ is the random complex amplitude, $a(t) = |\alpha(t)|$ and φ is overall the random phase. Argue that for N large, the probability of a given complex amplitude is Gaussian distributed in amplitude, and independent of phase

$$p(\alpha(t)) = \frac{1}{\pi N} e^{-|\alpha(t)|^2/N}$$

(c) Argue (filling in details from lecture) that under the *ergodic assumption* (the random signal samples different values according to the given probability distribution, so ensemble averages equal time averages), the two-time correlation function is

$$\left\langle E^*(t)E(t+\tau)\right\rangle = NE_0^2 e^{-i\omega\tau} e^{-\tau/\tau_0}$$

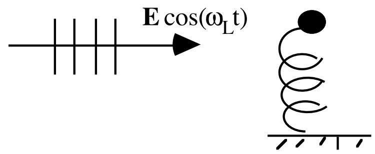
(d) The electric field, on average is zero, but there are fluctuations around the average. This implies that the intensity $I(t) = |E(t)|^2$ fluctuates. Using $E(t) = \sum_{i=1}^{N} E_0 e^{-i\omega t} e^{i\phi_i(t)}$, show that

$\left\langle I(t)\right\rangle = NE_0^2$, $\left\langle (I(t))^2\right\rangle = 2\left\langle I\right\rangle^2 \Rightarrow \Delta I = \left\langle I\right\rangle$, and generally the probability distribution of intensities is

$$P[I(t)] = \frac{1}{\left\langle I\right\rangle} e^{-\frac{I(t)}{\langle I\rangle}} \Rightarrow \left\langle (I(t))^n\right\rangle = n!\left\langle I\right\rangle^n$$

Problem 3: Lorentz oscillator model of scattering (10 points)

Consider the scattering of an electromagnetic wave by a damped Lorentz oscillator



(a) The absorption cross section, σ_{abs}, is defined as the rate at which energy is absorbed by an atom, divided by the incident flux of energy, the intensity $I = \frac{c}{8\pi}|\mathbf{E}_0|^2$ (CGS units). Show that the classical model of absorption gives,

$$\sigma_{abs,class} = \frac{2\pi^2 e^2}{mc} g(\omega_L)\,, \text{ where } g(\omega) = \frac{\Gamma_{rad}/(2\pi)}{(\omega-\omega_{eg})^2 + \Gamma_{rad}^2/4} \text{ is the line shape.}$$

Assume near resonance so that $\Delta = \omega_L - \omega_0 << \omega_L, \omega_0$.

(b) In the case of radiative damping, all energy absorbed is re-radiated, and is thus *scattered*. Use standard scattering theory to derive the differential scattering cross section for the Lorentz oscillator model, $\frac{d\sigma_{scat}}{d\Omega}$, and after integrating over all solid angles, show that the total scattering cross section equals the absorption cross section found in part (a). Here take $\Gamma_{rad} = \frac{2}{3}\frac{e^2}{mc^3}\omega_0^2$.

(c) We can re-derive the expression for the classical natural linewidth Γ_{rad} that we found in class via radiation reaction by looking directly at energy conservation in the scattering process. For the field on resonance, equate the time averaged absorbed power (rate at which field does work on electron, averaged over a period of oscillation) to the Lamor formula for the averaged radiated power to show,

$$\Gamma_{rad} = \frac{2}{3}\frac{e^2}{mc^3}\omega_0{}^2 = \frac{2}{3}\left(k_0 r_c\right)\omega_0\,, \text{ where } r_c \text{ is the classical electron radius.}$$

Evaluate this for the case the sodium “D2 resonance” (the yellow light in stree light), of excitation wavelength is 589 nm. The quantum decay rate is $\Gamma/2\pi = 9.8$ MHz. What is the oscillator strength of the transition?

(d) Show that the scattering cross section can be rëexpressed as $\sigma_{scat} = \frac{6\pi \lambdabar_0^2}{1+\left(4\Delta^2/\Gamma_{rad}^2\right)}$, where $\lambdabar_0 = \lambda/2\pi$. This expression holds true quantum mechanically as well with $\Gamma_{rad} \to \Gamma$.